First-Order Logical Limit Laws, Ordered Structures, and Permutation Classes

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Overview

Introduction

Fix a first-order language $\mathcal{L}$ and a class $\mathcal{C}$ of finite (yet arbitrarily large) $\mathcal{L}$-structures. How does a randomly selected $\mathcal{C}$-structure of size n behave as n becomes infinitely large?

Introduction: zero-one laws

Definition

A class $\mathcal{C}$ of first-order structures admits a **zero-one law** if, for any $\mathcal{L}$-sentence φ, the probability that a randomly selected $\mathcal{C}$-structure of size n satisfies φ converges asymptotically to zero or one.

Introduction: zero-one laws

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Finite graphs, expressed in $\mathcal{L} = \{E\}$, are a classical example.

Introduction: logical limit laws

Definition

A class $\mathcal{C}$ of first-order $\mathcal{L}$-structures admits a **logical limit law** if, for any sentence φ, the probability that a randomly selected $\mathcal{C}$-structure of size n satisfies φ *converges* asymptotically (not necessarily to zero or one).

We distinguish between labeled and unlabeled limit laws.

- **Labeled:** count all possible structures
- **Unlabeled:** count structures up to isomorphism

Main Results

Theorem

Convex linear orders and layered permutations admit both unlabeled and labeled limit laws. Compositions admit an unlabeled limit law.

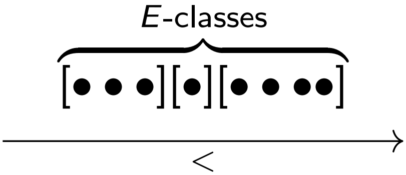
Convex linear orders

Definition

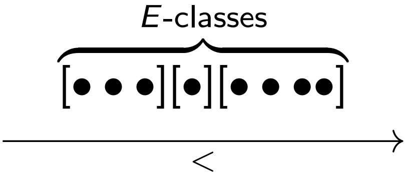
Let $\mathcal{L}$ be the language containing two binary relations: $<$ and E. A **convex linear order** is an $\mathcal{L}$-structure where:

- $<$ is a total order on points
- E is an equivalence relation
- $x\,E\,z,\ x < y < z \Rightarrow z\,E\,x, y$

Convex linear orders



Convex linear orders



From this point forward, we work over the domain $[n] = \{1, 2, \ldots, n\}$ for arbitrarily large n. The convex linear order with one point will be denoted by $\bullet$.

Sum operations

Definition

Let $\mathfrak{C}, \mathfrak{D}$ be convex linear orders.

- $\widehat{\mathfrak{C}}$ is the convex linear order obtained by adding one point to the last class of $\mathfrak{C}$
- $\mathfrak{C} \oplus \mathfrak{D}$ is the convex linear order which places $\mathfrak{D}$ $<$-after $\mathfrak{C}$.

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$$\widehat{[\bullet\bullet][\bullet]} = [\bullet\bullet][\bullet\bullet]$$

$$[\bullet\bullet][\bullet] \oplus [\bullet] = [\bullet\bullet][\bullet][\bullet]$$

Convex linear orders

Lemma

Every finite convex linear order of size n can be uniquely constructed by applying $\widehat{(-)}$ and/or $- \oplus \bullet$ to $\bullet$ repeatedly. This is done in $n-1$ steps.

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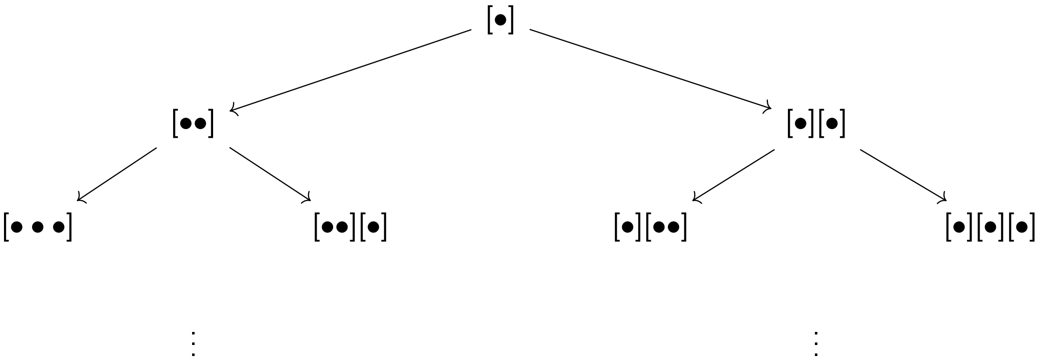
Proof

Proceed by induction.

- $n = 1$, trivial
- When $n = 2$, two possible cases: $\mathfrak{C} \simeq \bullet \oplus \bullet$ or $\mathfrak{C} \simeq \widehat{\bullet}$
- In general: last class of $\mathfrak{C}$ contains one or more points. Apply $- \oplus \bullet$ or $\widehat{(-)}$ respectively.

□

Convex linear orders



Ehrenfeucht–Fraïssé games

- Ehrenfeucht–Fraïssé game on two structures: back-and-forth game between players **Spoiler** and **Duplicator** in which corresponding points are marked on each structure
- In game of length k between $\mathfrak{A}$ and $\mathfrak{B}$, Duplicator has a winning strategy iff $\mathfrak{A}$ and $\mathfrak{B}$ agree on all sentences of quantifier depth at most k. Write $\mathfrak{A} \equiv_k \mathfrak{B}$ in this case.

Equivalence lemmas

Lemma

Let $\mathfrak{M}, \mathfrak{N}, \mathfrak{M}', \mathfrak{N}'$ be convex linear orders such that $\mathfrak{M} \equiv_k \mathfrak{N}$ and $\mathfrak{M}' \equiv_k \mathfrak{N}'$. The following equivalences hold:

- $\mathfrak{M} \oplus \mathfrak{M}' \equiv_k \mathfrak{N} \oplus \mathfrak{N}'$
- $\widehat{\mathfrak{M}} \equiv_k \widehat{\mathfrak{N}}$

Equivalence lemmas

Lemma

For a convex linear order $\mathfrak{M}$ and $k \in \mathbb{N}$, there exists $\ell \in \mathbb{N}$ such that for all $s, t > \ell$,

$$\bigoplus_{s} \mathfrak{M} \equiv_k \bigoplus_{t} \mathfrak{M}$$

Extension of sum operations

Extend sum operations to equivalence classes:

$$C \oplus \bullet := [\mathfrak{M} \oplus \bullet]_{\equiv_k}$$

$$\widehat{C} := \left[\widehat{\mathfrak{M}}\right]_{\equiv_k}$$

The limit law

General idea:

- For a first-order sentence φ having quantifier rank k, construct a Markov chain M_φ
- States of M_φ are $\equiv_k$-classes
- Probability of a randomly-selected structure of size n satisfying φ is probability that M_φ is in a state satisfying φ after n transitions
- Finite linearly-ordered structures are rigid, so no distinction between labeled and unlabeled limit laws in this case

Constructing a Markov chain

Definition

A Markov chain is **fully aperiodic** if there do *not* exist disjoint sets of M-states $P_0, P_1, \ldots, P_{d-1}, d > 1$ such that, for every state in P_i, the chain M transitions to a state in P_{i+1} with probability 1 (and P_{d-1} transitions to P_0).

Lemma

Let M be a finite, fully aperiodic Markov chain with initial state S, and let $Pr^{n-1}(S, Q)$ denote the probability that M is in state Q after $n - 1$ steps. For any choice of Q, $\lim_{n\to\infty} Pr^{n-1}(S, Q)$ converges.

Constructing a Markov chain

Suppose φ is an $\mathcal{L}$-sentence having quantifier depth k. We construct a Markov chain M_φ as follows:

- Starting state : $[\bullet]_{\equiv_k}$
- From any $\equiv_k$-class C, there are two possible transitions: to $C \oplus \bullet$ or $\widehat{C}$
- Each transition probability is $1/2$

The limit law

Theorem

M_φ is fully aperiodic for any first-order sentence φ.

Proof

Suppose it were not.

- There would exist disjoint sets of M_φ-states $P_0, P_1, \ldots P_{d-1}$ forming a cycle
- For any $Q \in P_0$, $Q \oplus i\bullet$ is in P_0 iff $d \mid i$
- By earlier equivalence lemmas, $Q \oplus i\bullet \equiv_k Q \oplus (i+1)\bullet$ for sufficiently large i

□

The limit law

Theorem

Convex linear orders admit a logical limit law.

Proof

Fix a first-order sentence φ, and consider M_φ.

- For every state of M_φ, either every structure in the state satisfies φ or no structures do
- Let S_φ denote the set of states in M_φ for which all structures in that state satisfy φ.
- $\widehat{(-)}$ and $- \oplus \bullet$ are well-defined on $\equiv_k$-classes. Hence, moving $n-1$ steps in M_φ is equivalent to starting with any structure in the current state, applying $\widehat{(-)}$ or $- \oplus \bullet$ as needed, and taking the $\equiv_k$-class.

The limit law

Proof (continued)

- The probability that after n steps, M_φ is in a state of S_φ equals probability that a uniformly randomly selected structure of size n satisfies φ
- Suffices to show that $\lim_{n\to\infty} \sum_{Q\in S_\varphi} Pr^{n-1}(\bullet, Q)$ converges, which follows from aperiodicity

$\square$

Transfer lemma

Fix languages $\mathcal{L}_0, \mathcal{L}_1$ and classes $\mathcal{C}_0, \mathcal{C}_1$ of $\mathcal{L}_0, \mathcal{L}_1$ structures respectively.

Lemma

Let f be a map from the set of $\mathcal{L}_0$-structures to the set of $\mathcal{L}_1$-structures, and g a map from the set of $\mathcal{L}_0$-sentences to the set of $\mathcal{L}_1$-sentences such that, for any $\mathcal{C}_0$-structure $\mathfrak{M}$ and $\mathcal{L}_0$-sentence φ:

1. $\mathfrak{M} \models \varphi \iff f(\mathfrak{M}) \models g(\varphi)$
2. f is bijective between structures of size n for all n
3. The class $\mathcal{C}_1$ admits a limit law

Then, $\mathcal{C}_0$ also admits a limit law.

Transfer lemma

Proof

Let φ be an $\mathcal{L}_0$ sentence and a_0 the number of size n structures in $\mathcal{C}_0$ satisfying φ. Likewise, let a_1 be the number of size n structures in $\mathcal{C}_1$ satisfying $g(\varphi)$. For a randomly selected $\mathcal{C}_0$-structure $\mathfrak{M}$ (of size n),

$$Pr(\mathfrak{M} \models \varphi) = \frac{a_0}{|\mathcal{C}_0|}$$

$$Pr(f(\mathfrak{M}) \models g(\varphi)) = \frac{a_1}{|\mathcal{C}_1|}$$

From bijectivity of f, $|\mathcal{C}_0| = |\mathcal{C}_1|$, and by (1), $a_1 = a_0$. Thus, the probabilities are equal for any φ, with the second one convergent. This gives a limit law for $\mathcal{C}_0$. $\square$

Uniform interdefinability

Definition

Classes $\mathcal{C}_0, \mathcal{C}_1$ of structures over a common finite domain are **uniformly interdefinable** if there exists a bijection on structures $f_I : \mathcal{C}_0 \to \mathcal{C}_1$, along with formulae $\varphi_{R_{0,i}}, \varphi_{R_{1,i}}$ for each relation $R_{0,i}$ in $\mathcal{L}_0$ and $R_{1,i}$ in $\mathcal{L}_1$ such that, for each $\mathfrak{M}_0$ in $\mathcal{C}_0$ and $\mathfrak{M}_1$ in $\mathcal{C}_1$:

- $\mathfrak{M}_0 \models R_{0,i}(\bar{x}) \iff f_I(\mathfrak{M}_0) \models \varphi_{R_{0,i}}(\bar{x})$
- $\mathfrak{M}_1 \models R_{1,i}(\bar{x}) \iff f_I^{-1}(\mathfrak{M}_1) \models \varphi_{R_{1,i}}(\bar{x})$

Uniform interdefinability

Theorem

Let $\mathcal{C}_0$, $\mathcal{C}_1$ be uniformly interdefinable classes of $\mathcal{L}_0$, $\mathcal{L}_1$ structures. If $\mathcal{C}_1$ admits a logical limit law, $\mathcal{C}_0$ admits one as well.

Proof

Apply the transfer lemma. Take the transfer maps f, g to be:

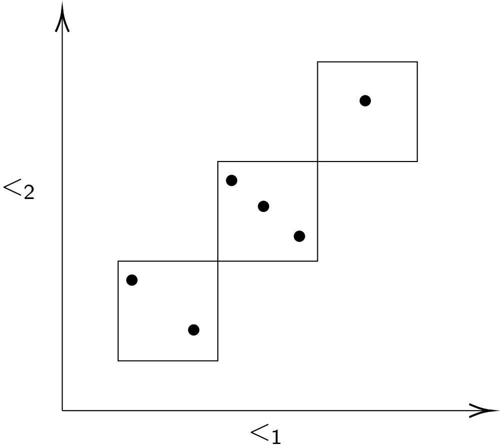
- $f = f_I$
- g is the map sending an $\mathcal{L}_0$-sentence to the $\mathcal{L}_1$-sentence with each occurrence of $R_{0,i}$ replaced with $\varphi_{R_{0,i}}$

$\square$

Layered permutations

- Permutations can be viewed as structures in the language $\mathcal{L} = \{<_1, <_2\}$ with two linear orders. The order $<_1$ gives the unpermuted order of the points, and $<_2$ describes the points after applying the permutation.
- The class of all permutations *cannot* admit a limit law, but certain subclasses can
- **Blocks** are maximal subsets which are monotone $<_1/<_2$-intervals
- A **layered permutation** is composed of increasing blocks, each containing a decreasing permutation

Layered permutations



Interdefinability with convex linear orders

Lemma

Layered permutations and convex linear orders are uniformly interdefinable.

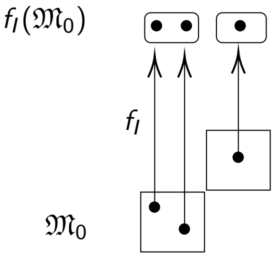
Proof

Define f_l to be the map taking blocks of a layered permutation to classes of a convex linear order, and points in an order-preserving manner. The relations $<_1$ and $<_2$ are rewritten as:

- $\varphi_{<_1} : a <_1 b \rightsquigarrow a < b$
- $\varphi_{<_2} : a <_2 b \rightsquigarrow (a \; E \; b \wedge b < a) \vee (\neg(a \; E \; b) \wedge a < b)$

$\square$

Interdefinability with convex linear orders



The limit law

Theorem

Layered permutations admit a logical limit law.

Proof

Layered permutations are uniformly interdefinable with convex linear orders. Because convex linear orders admit a logical limit law, layered permutations admit one as well. □

1. Introduction
2. Convex Linear Orders
3. Uniform Interdefinability
4. Layered Permutations
5. **Compositions**
6. Further Work and Questions

Fractured orders

- Let $\mathcal{L}_0 = \{E, <\}$ be the language of convex linear orders
- Define $\mathcal{L}_1 = \{E, \prec_1, \prec_2\}$
- Fractured orders take a convex linear order $<$ and break it into two parts: $\prec_1$ *between* E-classes, and $\prec_2$ *within* E-classes.

Fractured orders

Definition

A **fractured order** is an $\mathcal{L}_1$-structure where:

1. $\prec_1$, $\prec_2$ are partial orders
2. E is an equivalence relation
3. Distinct points a, b are $\prec_1$-comparable iff they **are not** E-related
4. Distinct points a, b are $\prec_2$-comparable iff they **are** E-related
5. $a \, E \, a'$, $a \prec_1 b \Rightarrow a' \prec_1 b$ (convexity)

Denote the class of all finite fractured orders by $\mathcal{F}$.

Fractured orders

Theorem

Fractured orders and convex linear orders are uniformly interdefinable.

Proof

Define $f_l : \mathcal{F} \to \mathcal{C}_0$ such that:

- $\mathfrak{M}_1 \models a\ E\ b \iff f_l(\mathfrak{M}_1) \models a\ E\ b$
- $\mathfrak{M}_1 \models a \prec_1 b \iff f_l(\mathfrak{M}_1) \models \neg a\ E\ b \wedge a < b$
- $\mathfrak{M}_1 \models a \prec_2 b \iff f_l(\mathfrak{M}_1) \models a\ E\ b \wedge a < b$

This map satisfies the requirements for uniform interdefinability. □

Reducts and limit laws

Lemma

Let $\mathcal{L}$ be a language and $\mathcal{L}' \subset \mathcal{L}$. Given a class $\mathcal{C}$ of $\mathcal{L}$-structures which admits a logical limit law, any class $\mathcal{C}'$ of $\mathcal{L}'$-structures which expand uniquely to $\mathcal{C}$-structures also admits a logical limit law.

Reducts and limit laws

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Proof

Construct the transfer maps f and g from earlier:

- f is taken to be the map sending a structure in $\mathcal{C}'$ to its unique expansion in $\mathcal{C}$
- This expansion is unique, hence f is bijective on structures of size n for all n
- g is given by the identity map on formulas

□

Compositions

- **Compositions** are structures in the reduct $\mathcal{L}_2 \subset \mathcal{L}_1$ given by $\mathcal{L}_2 = \{E, \prec_1\}$
- Order defined on equivalence classes, but not on points within each class

Compositions

Lemma

Every composition expands uniquely to a fractured order, up to isomorphism.

Proof

There is a unique way to linearly order each E-class individually. Because ordering these classes determines $\prec_2$, there is a unique way to define $\prec_2$ on any composition, expanding it to a fractured order. □

Compositions

Theorem

The class of compositions admit an unlabeled logical limit law.

Proof

The language of compositions is a reduct of the language of fractured orders, and every composition expands uniquely to a fractured order. The class of fractured orders admits a logical limit law, therefore, by the previous lemma, compositions admit a limit law as well. □

Questions

- Do compositions admit a labeled limit law?
- Which other classes of permutations admit a limit law?
    - 231-avoiding permutations [1]
    - Random permutations following a Mallows distribution [3]
- Can further analogues of the $\oplus$ operator be extended to show limit laws for other classes of ordered structures?

References

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